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ON THE CHARACTERIZATION OF THE DIMENSION OF A COMPACT METRIC SPACE K BY THE REPRESENTING MATRICES OF $C(K)$

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ABSTRACT

We state and prove some characterizations of the topological dimension of compact metric spaces K by the matrices representing $C(K)$ as a predual of $L_1(\mu)$.

1. Introduction

In [5] Lazar and Lindenstrauss have introduced the concept of representing matrices for separable preduals of L_1. They show, that if X is a separable Banach space, such that X^* is isometric to $L_1(\mu)$ for some measure μ, then X has a representation

(1) $$X = \overline{\bigcup_{n=1}^{\infty} E_n}, \ E_n \subset E_{n+1}, \ E_n = l_n^\infty, \ n = 1, 2, \cdots.$$

Moreover, the isometries $E_n \to E_{n+1}$ can be chosen so that if $\{e_{n,i}\}_{i=1}^n$ is the unit-vector basis of l_n^∞, then there exist reals $\{a_{n,i}\}_{i=1}^n$, so that

(2) $$e_{n,i} = e_{n+1,i} + a_{n,i} e_{n+1,n+1}; \ 1 \leqq i \leqq n; \ n = 1, 2, \cdots,$$

and

(3) $$\sum_{i=1}^{n} | a_{n,i} | \leqq 1, \ n = 1, 2, \cdots.$$

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The triangular matrix $A = \{a_{n,i}\}_{i \leq n}$ is called a representing matrix of X. Theorem 5.2 of [5] implies, that if X is the space $A(S)$ of continuous affine functions on a compact metric Choquet-simplex S, then A can be chosen so that

$$\sum_{i=1}^{n} a_{n,i} = 1, \quad n = 1, 2, \cdots. \tag{4}$$

In particular (4) applies to spaces $X = C(K)$ of real-valued continuous functions on a compact metric space K.

Theorem 5.1 of [5] states that $X = C(K)$ for a compact metric 0-dimensional space K if and only if there exists a representing matrix A for X, so that, in addition to (4), A has the property, that for each n there exists an $1 \leqq i \leqq n$ with $a_{n,i} = 1$.

In [7, p. 167] an extension of this theorem was conjectured. We present an example which shows that the answer to the problem in [7] is negative, but prove an extension very similar to that proposed in [7]. Let now $A = \{a_{n,i}\}_{i \leq n}$ be a matrix, for which (3) and (4) are satisfied. For each $n \geqq 1$ and $1 \leqq i \leqq n$ we define inductively a sequence $\{P^l_{n,i}\}_{l=1}^{\infty}$ as follows:

$$P^l_{n,i} = \delta_{i,l} \quad \text{for } l = 1, \cdots, n \tag{5}$$

and

$$P^l_{n,i} = \sum_{j=1}^{l-1} a_{l-1,j} P^j_{n,i} \quad \text{for } l = n+1, n+2, \cdots. \tag{6}$$

Observe that

$$P^{n+1}_{n,i} = a_{n,i} \quad \text{and} \tag{7}$$

$$\sum_{i=1}^{n} P^l_{n,i} = 1 \quad \text{for } n, l = 1, 2, \cdots. \tag{8}$$

We define also the real number $\lambda(A)$ by

$$\lambda(A) = \limsup_{n \to \infty} \inf_{l \geq 1} \max_{1 \leq i \leq n} P^l_{n,i}. \tag{9}$$

The next section is devoted to the investigation of the rôle played by the $P^l_{n,i}$ and $\lambda(A)$ in the case where X is a $C(K)$-space.

2. Preliminary observations on representing matrices of $C(K)$ spaces

Throughout this section we assume that $A = \{a_{n,i}\}_{i \leq n}$ is a matrix satisfying (3) and (4) and representing a $C(K)$-space for a compact metric K. Let

$\{e_{n,i}\}_{i \le n, n=1,2,\cdots}$ be the sequence of unit-vector bases of the l_n^∞ corresponding to A by (1) and (2). As observed in [5, p. 185], $e_{1,1}$ is an extreme point in the unit ball of $C(K)$, so $|e_{1,1}(x)| = 1$ for all $x \in K$, and (passing to $e_{n,i}^1 = e_{n,i} \cdot e_{1,1}$) we may assume that $e_{1,1} \equiv 1$. Hence, by (2) and (4), the sets $\{e_{n,i}\}_{i \le n}$, $n = 1, 2, \cdots$ are non-negative partitions of unity with $\| e_{n,i} \| = 1$. Set

$$H(e_{n,i}) = \{x \in K \mid e_{n,i}(x) = 1\}, \quad i \le n, \quad n = 1, 2, \cdots, \tag{10}$$

Each $H(e_{n,i})$ is a non-empty compact subset of K and by (2) we have $H(e_{n,i}) \supset H(e_{n+1,i})$, so for each i, $\bigcap_{n=i}^\infty H(e_{n,i}) \neq \emptyset$.

Lemma 1. *For each $i = 1, 2, \cdots$ $\bigcap_{n=i}^\infty H(e_{n,i})$ consists of a single point.*

Proof. Assume $x, y \in \bigcap_{n=i}^\infty H(e_{n,i})$ and $x \neq y$. Let $f \in C(K)$, so that $f(x) = 1$ and $f(y) = 0$. By (1) there exist an $n \ge i$ and a $g = \sum_{j=1}^n \alpha_j e_{n,j} \in E_n$, so that $\| f - g \| < \frac{1}{2}$. By assumption, $e_{n,i}(x) = e_{n,i}(y) = 1$, so $e_{n,j}(x) = e_{n,j}(y) = 0$ for $i \neq j$. Hence $g(x) = g(y) = \alpha_i$ and $|\alpha_i| = |f(y) - g(y)| < \frac{1}{2}$, but $|1 - \alpha_i| = |f(x) - g(x)| < \frac{1}{2}$, which is a contradiction. ■

Put, in view of Lemma 1,

$$\{x_i\} = \bigcap_{n=i}^\infty H(e_{n,i}) \quad i = 1, 2, \cdots. \tag{11}$$

Lemma 2. *The set $\{x_i\}_{i=1}^\infty$ is dense in K.*

Proof. Assume the converse, i.e. there exists an open subset U of K with $\{x_i\}_{i=1}^\infty \cap U = \emptyset$. Let $f \in C(K)$ be such that $\| f \| = 1$ and $f(K \setminus U) = \{0\}$. By (1) there exist an n and a $g = \sum_{i=1}^n \alpha_i e_{n,i} \in E_n$ with $\| f - g \| < \frac{1}{2}$. As in Lemma 1 we get that $\alpha_i = g(x_i)$, so since $f(x_i) = 0$, $|\alpha_i| < \frac{1}{2}$. Hence $|g| \le \sum_{i=1}^n |\alpha_i| e_{n,i} < \frac{1}{2} \cdot 1_K$ or $\| g \| < \frac{1}{2}$. This gives that $1 = \| f \| \le \| f - g \| + \| g \| < \frac{1}{2} + \frac{1}{2} = 1$. ■

Lemma 3. $P_{n,i}^l = e_{n,i}(x_l)$, $i \le n$, n, $l = 1, 2, \cdots$.

Proof. It is easily checked that the sequence $\{e_{n,i}(x_l)\}_{l=1}^\infty$ satisfies (5) and (6). ■

From (9) and Lemma 3 it follows that

$$\lambda(A) = \limsup_{n \to \infty} \inf_{1 \le l} \max_{1 \le i \le n} e_{n,i}(x_l). \tag{12}$$

Hence, for each $\lambda < \lambda(A)$, there exist infinitely many n's, so that

$$\inf_{1 \le l} \max_{1 \le i \le n} e_{n,i}(x_l) \ge \lambda. \tag{13}$$

This and Lemma 2 prove

LEMMA 4. *For each $\lambda < \lambda(A)$, there exist infinitely many n's, so that* $\max_{1 \le i \le n} e_{n,i} \geqq \lambda$.

For each $n = 1, 2, \cdots$ define the projection $Q_n : C(K) \to E_n$ by

$$\forall f \in C(K): Q_n f = \sum_{i=1}^{n} f(x_i)\, e_{n,i}. \tag{14}$$

Clearly $\| Q_n \| = 1$ and Q_n maps $C(K)$ onto E_n. We claim that

LEMMA 5. *For all $f \in C(K)$, $\lim_{n \to \infty} \| f - Q_n f \| = 0$.*

PROOF. Let $f \in C(K)$ and $\varepsilon > 0$. By (1) there exists an n_0, so that for all $n \geqq n_0$ there is a $g = \sum_{i=1}^{n} g(x_i)\, e_{n,i} \in E_n$ with $\| f - g \| < \varepsilon$. In particular $| g(x_i) - f(x_i) | < \varepsilon$, so if $x \in K$, $| g(x) - Q_n f(x) | \leqq \sum_{i=1}^{n} | g(x_i) - f(x_i) |\, e_{n,i}(x) \leqq \varepsilon$. This gives that $\| g - Q_n f \| \leqq \varepsilon$, so for all $n \geqq n_0$, $\| f - Q_n f \| \leqq \| f - g \| + \| g - Q_n f \| \leqq 2\varepsilon$. ∎

3. Statement of the main results

In [7] Lindenstrauss and Tzafriri proposed the following extension of [5, theor. 5.1]:

PROBLEM 1. Let K be a compact metric space. Is it true that $\dim K \leqq d$ if and only if $C(K)$ can be represented by a matrix $A = \{a_{n,i}\}_{i \le n}$ with $\sum_{i=1}^{n} a_{n,i} = 1$ for all n and so that for each n at most $d + 1$ of the numbers $\{a_{n,i}\}_{i=1}^{n}$ are non-zero?

(By $\dim K$ we denote the topological dimension of K as defined in e.g. [9] or [4].) We can prove the following two theorems which characterize $\dim K$ by the matrices representing $C(K)$. Theorem 2 gives an affirmative answer to the "only if"—part of Problem 1, but in Section 8 we present an example which shows that the property in Problem 1 eventually does not say anything about $\dim K$, when $d \geqq 1$.

THEOREM 1. *Let K be a compact metric space. Then $\dim K \leqq d$ if and only if $C(K)$ can be represented by a matrix $A = \{a_{n,i}\}_{i \le n}$ with $\sum_{i=1}^{n} a_{n,i} = 1$ for all n and so that*

$$\lambda(A) > \frac{1}{d+2}. \tag{15}$$

THEOREM 2. *Let K be a compact metric space. Then $\dim K \leqq d$ if and only if $C(K)$ can be represented by a matrix $A = \{a_{n,i}\}_{i \le n}$ with $\sum_{i=1}^{n} a_{n,i} = 1$ for all n and enjoying the following properties:*

(16) *for each n, at most d + 1 of the numbers* $\{a_{n,i}\}_{i=1}^n$ *are non-zero, and all the non-zero* $a_{n,i}$ *are equal (and hence equal to* $1/J_{n-1}$, *if* J_{n-1} *is the number of non-zero* $a_{n,i}$*). In particular* $a_{n,i} \in \{0, 1/(d+1), 1/d, \cdots, 1\}$.

(17) *for infinitely many n's and all l, at most d + 1 of the numbers* $\{P^l_{n,i}\}_{i=1}^n$ *are non-zero.*

In particular for these n's, $\max_{1 \le i \le n} P^l_{n,i} \geqq 1/(d+1)$ *for all l, so* $\lambda(A) \geqq 1/(d+1)$.

REMARKS. (i) Since (17) implies (15) it clearly suffices to prove the "if"-part and the "only if"-part of Theorems 1 and 2 respectively. (ii) We can construct a matrix representing $C(0,1)$ with the property (16) but not satisfying (17). (iii) By Lemmas 3 and 4 and by (7), (16) only states that in the point x_{n+1} at most $d+1$ of the functions $\{e_{n,i}\}_{i=1}^n$ are non-zero, whereas (17) implies that for infinitely many n's, $\{e_{n,i}\}_{i=1}^n$ has the property that in each point of K at most $d+1$ members of this set are non-zero.

An immediate consequence of Theorems 1 and 2 is

THEOREM 3. *Let K be a compact metric space. Then*

(18)
$$\dim K = \frac{1}{\max \lambda(A)} - 1$$

where the max *is taken over all representing matrices of* $C(K)$, *satisfying* (4). *In particular* $\dim K = \infty$ *if and only if* $\lambda(A) = 0$ *for every A representing* $C(K)$.

In the next section we recall some theorems from dimension theory, which we shall use in the sequel. In Section 5 we prove the "only if"-part of Theorem 2 and in Section 6 we prove the "if"-part of Theorem 1. In Section 7 we state and prove an extension of the "if"-part of Theorem 1 to the setting of Choquet-simplices. The final section is devoted to examples and open problems.

4. Facts from dimension theory

Let K be a compact metric space. An open cover $\mathcal{U}$ of K is a finite collection of open subsets of K whose union is K. By mesh $\cdot\,\mathcal{U}$ we denote $\max_{U \in \mathcal{U}}$ diameter (U). It is easy to prove and well known, that if $\mathcal{U}$ is an open cover of K, then there exists a $\delta > 0$ called a Lebesgue number of $\mathcal{U}$ such that each subset of K with a diameter less than δ is contained in some element of $\mathcal{U}$. $\mathcal{U}$ is said to be of order $\leqq d$ if no $d+2$ distinct members of $\mathcal{U}$ intersect. The following characterization of dimension is proved in [4]: $\dim K \leqq d$ if and only if for each $\varepsilon > 0$ there exists an open cover $\mathcal{U}$ of K with mesh $\mathcal{U} \leqq \varepsilon$ and of order less than or equal to d.

The following deep theorem is due to Nagata ([9, p. 138]). See also [8] for a proof. Nagata's theorem will be our main tool in proving (17) of Theorem 2.

NAGATA'S THEOREM. *Let K be a compact metric space. Then $\dim K \leqq d$ if and only if there exists a topology preserving metric ρ on K (called a Nagata d-dimensional metric) with the following property:*

(19) *for each $\varepsilon > 0$ and every $d+3$ points $y_1, \cdots, y_{d+2}, x$ in K with $\rho(S(x, (\varepsilon/2)), y_i) < \varepsilon$, $i = 1, \cdots, d+2$, there exist $1 \leqq i < j \leqq d+2$ such that $\rho(y_i, y_j) < \varepsilon$.*

($S(x, r)$ denotes the ball in K with center x and radius r). It is obvious that if ρ is a d-dimensional Nagata metric on K, then ρ has the following property:

(20) *if* $y_1, \cdots, y_{d+2}, x$ are points in K, then there exist i, j, k with $i \neq j$, so that $\rho(y_i, y_j) \leqq \rho(x, y_k)$.

Observe that the usual metric on the real line enjoys property (20), but not (19).

5. Proof of "only if"-part of Theorem 2

This proof is vitally influenced by the construction of the usual Schauder basis of $C(0, 1)$ (see [10, p. 11]). The reader is strongly advised to have the case $K = [0, 1]$ in mind when reading the proof.

Let $\dim K \leqq d$ and let ρ be a d-dimensional Nagata metric on K, in accordance with Nagata's theorem. We select now a sequence in K which will eventually play the rôle of that defined in (11). In the usual construction of the Schauder basis of $C(0, 1)$, this sequence is the set of dyadics.

First, let $\delta_1 = \text{diameter}(K)$ and let $\{x_1, \cdots, x_{n_1}\}$ be points in K so that

$$\rho(x_i, x_j) = \delta_1 \quad \text{for} \quad i \neq j, \tag{21}$$

$$\{S(x_i, \delta_1)\}_{i=1}^{n_1} \quad \text{is an open cover of } K. \tag{22}$$

Clearly $2 \leqq n_1 < \infty$. Next, let δ be a Lebesgue number of the cover mentioned in (22) and set

$$\delta_2 = \min\left\{\frac{1}{5}\delta,\ \max_{x \in K} \rho(x, \{x_1, \cdots, x_{n_1}\})\right\} \tag{23}$$

and pick $x_{n_1+1},\cdots,x_{n_2}$ in K so that

(24) $$\rho(x_i,x_j)\geqq\delta_2 \quad \text{for} \quad 1\leqq i<j\leqq n_2$$

and

(25) $$\{S(x_i,\delta_2)\}_{i=1}^{n_2} \quad \text{is an open cover of } K.$$

Continuing in this way we get a sequence $\{x_i\}_{i=1}^{\infty}$ in K, an increasing sequence $2\leqq n_1<n_2<\cdots$ of integers and positive reals $\delta_1,\delta_2,\cdots$ so that for each $l=1,2,\cdots$

(26) $$\rho(x_i,x_j)\geqq\delta_l \quad \text{for} \quad 1\leqq i<j\leqq n_l,$$

(27) $$\{S(x_i,\delta_l)\}_{i=1}^{n_l} \quad \text{is an open cover of } K,$$

(28) $$5\delta_{l+1} \quad \text{is a Lebesgue number of the covering} \quad \{S(x_i,\delta_l)\}_{i=1}^{n_l},$$

(29) $$\delta_{l+1}\leqq\max_{x\in K}\rho(x,\{x_1,\cdots,x_{n_l}\}).$$

(29) ensures only that $n_l<n_{l+1}$. Observe that $\Sigma_{r=l+1}^{\infty}\delta_r<\delta_l$ for each l and that $\{x_i\}_{i=1}^{\infty}$ is dense in K.

LEMMA 6. *$\{S(x_i,\delta_l)\}_{i=1}^{n_l}$ is of order $\leqq d$ for $l=1,2,\cdots$.*

PROOF. Let $x\in K$ and $x\in\bigcap_{j=1}^{d+2}S(x_{i_j},\delta_l)$. By (20) there exist a, b, c with $a\neq b$, so that $\rho(x_{i_a},x_{i_b})\leqq\rho(x_{i_c},x)<\delta_l$. This and (26) give that $i_a=i_b$. ∎

For convenience we introduce the following notation: we call the points $\{x_1,\cdots,x_{n_l}\}$ the lth generation, $l=1,2,\cdots$ For every integer $n>0$ there is a unique integer $l(m)$ such that $n_{l(m)}<m\leqq n_{l(m)+1}$. (If $1\leqq m\leqq n_1$, put $l(m)=0,n_0=0$.) For each $l=1,2,\cdots$ we define the relatives of x_m in the lth generation (l-rel's of x_m) as follows:

(30) if $l(m)<l$, the l-rel's of x_m are x_m itself,

(31) if $l(m)=l$, the l-rel's of x_m are those x_j in lth generation with $\rho(x_m,x_j)<\delta_l$,

(32) if $l(m)>l$, x_j is an l-rel of x_m if x_j is an l-rel of some $(l+1)$-rel of x_m.

This inductive definition can also be stated explicitly as follows:

(33) x_j is an l-rel of x_m if and only if either $l(m)<l$ and $j=m$, or $l\leqq l(m)$ and there exist a sequence of (not necessarily different) indices $j=j_l$, $j_{l+1},\cdots,j_{l(m)},j_{l(m)+1}=m$ with $j_i\leqq n_i$ and $\rho(x_{j_i},x_{j_{i+1}})<\delta_i$ for $i=l,l+1,\cdots,l(m)$.

Let us denote by R^l_m the set of relatives to x_m in the lth generation and by J^l_m the cardinality of R^l_m. We write also R_m and J_m for $R^{l(m)}_m$ and $J^{l(m)}_m$ respectively.

LEMMA 7. *If $x_j \in R^l_m$, then $\rho(x_m, x_j) < 2\delta_l$.*

PROOF. If $l > l(m)$ this is obvious, so let $l \leqq l(m)$. From (33) we get that $\rho(x_m, x_j) < \delta_l + \cdots + \delta_{l(m)} < \delta_l + \sum_{r=l+1}^{\infty} \delta_r < 2\delta_l$ (by 28). ■

LEMMA 8. *$J^l_m \leqq d + 1$ for all m and l.*

PROOF. If $l > l(m)$ this is obvious, so let $l \leqq l(m)$, and $x_j \in R^l_m$. By definition there is an $x_k \in R^{l+1}_m$, so that x_j is a relative to x_k in the lth generation. From Lemma 7 we get that $\rho(x_k, x_m) < 2\delta_{l+1} < \frac{1}{2}\delta_l$ (by 28), so $x_k \in S(x_m, \frac{1}{2}\delta_l)$. Since $\rho(x_j, x_k) < \delta_l$ this gives that $\rho(S(x_m, \frac{1}{2}\delta_l), x_j) < \delta_l$. If R^l_m consisted of more than $d+1$ points, we could apply Nagata's theorem to get x_{j_1} and x_{j_2} with $j_1 \neq j_2 \leqq n_l$ and $\rho(x_{j_1}, x_{j_2}) < \delta_l$. Since x_{j_1} and x_{j_2} are in the lth generation this would contradict (26). ■

REMARKS. (i) If $l = l(m)$, Lemma 8 follows easily from Lemma 7 and (20), so

(ii) if $K = [0, 1]$ and $\{x_i\}_{i=1}^{\infty}$ are the dyadics, Lemma 8 holds in spite of the observation following Nagata's theorem.

Let us now show that if x_n and x_m are close in the metric ρ on K, then they are also close in the sense that they have common relatives:

LEMMA 9. *Let n, m and l be positive integers. If $\rho(x_n, x_m) < \delta_{l+1}$ then there exists a common relative x^0 in the lth generation to all relatives of x_n and x_m in the $(l+1)$th generation, i.e.*

$$(34) \qquad \bigcap_{x_j \in R^{l+1}_m \cup R^{l+1}_n} R^l_j \neq \emptyset \quad \text{if} \quad \rho(x_n, x_m) < \delta_{l+1}.$$

PROOF. We claim that diameter$(R^{l+1}_m \cup R^{l+1}_n) < 5\,\delta_{l+1}$. Indeed, let $y, z \in R^{l+1}_m \cup R^{l+1}_n$. If $y, z \in R^{l+1}_n$ then by Lemma 7, $\rho(y, z) < \rho(y, x_n) + \rho(x_n, z) < 4\,\delta_{l+1}$. The same argument applies if $y, z \in R^{l+1}_m$. If finally $y \in R^{l+1}_n$ and $z \in R^{l+1}_m$, also by Lemma 7, $\rho(y, z) < \rho(y, x_n) + \rho(x_n, x_m) + \rho(x_m, z) < 5\,\delta_{l+1}$. From (28) it follows that there is an x^0 in the lth generation so that $R^{l+1}_m \cup R^{l+1}_n \subset S(x^0, \delta_l)$ and by definition of relatives we are done. ■

DEFINITION. For $l = 1, 2, \cdots$ let E_{n_l} be the subspace of $C(K)$ consisting of those functions for which

(35) $$\text{For all} \quad m > n_l : f(x_m) = \frac{1}{J_m} \sum_{x_j \in R_m} f(x_j).$$

(Recall that $R_m = R_m^{l(m)}$ by definition.)

REMARK. In the case $K = [0,1]$, E_{n_l} consists of those piecewise linear functions in $C(0,1)$ whose points of indifferentiability are contained in the set of dyadics in the lth generation, $\{0, 1/2^{l-1}, 2/2^{l-1}, \cdots, 1\}$.

LEMMA 10. *E_{n_l} is an n_l-dimensional subspace of $C(K)$. Moreover for every choice of n_l reals, $t_1, \cdots, t_{n_l}$, there exists an $f \in E_{n_l}$ with $f(x_i) = t_i$, $i = 1, \cdots, n_l$.*

PROOF. Clearly for every $f \in E_{n_l}$, $\| f \| = \sup\{ | f(x_i) | \mid 1 \leqq i \leqq n_l \}$, and hence $\dim E_{n_l} \leqq n_l$. We prove the second part of the lemma. Let $t_1, \cdots, t_{n_l}$ be given. Define a function f on $\{x_i\}_{i=1}^{\infty}$ by

(36) $$f(x_i) = t_i \quad \text{for} \quad 1 \leqq i \leqq n_l \quad \text{and} \quad f(x_m) = \frac{1}{J_m} \sum_{x_i \in R_m} f(x_i) \quad \text{for} \quad m > n_l.$$

This determines f uniquely on $\{x_m\}_{m=1}^{\infty}$. It remains to show that f is uniformly continuous on $\{x_m\}_{m=1}^{\infty}$ and hence can be extended (uniquely) to a function in $C(K)$. Set $a = \min_{1 \leqq i \leqq n_l} t_i$ and $b = \max_{1 \leqq i \leqq n_l} t_i$. The uniform continuity of f follows from

(37) $$\text{If} \quad \rho(x_n, x_m) < \delta_k \quad \text{then} \quad | f(x_n) - f(x_m) | \leqq \left(\frac{d}{d+1} \right)^{k-l} (b-a).$$

To prove (37), let k be given. We may assume that $k > l$. (For $k \leqq l$ (37) is trivial). Let x_n and x_m with $\rho(x_n, x_m) < \delta_k$ be given. Since $\delta_k \leqq \delta_{l+1}$ we can use Lemma 9 to find an x^l in $\bigcap_{x_j \in R_n^{l+1} \cup R_m^{l+1}} R_j^l$. Let $x_i \in R_n^{l+1} \cup R_m^{l+1}$. Then

(38) $$\begin{aligned} f(x_i) &= \frac{1}{J_i} \sum_{x_j \in R_i^l} f(x_j) \\ &= \frac{1}{J_i} f(x^l) + \frac{1}{J_i} \sum_{x_j \in R_i^l \setminus \{x^l\}} f(x_j) \\ &\leqq \frac{1}{J_i} f(x^l) + \frac{J_i - 1}{J_i} b \leqq \frac{f(x^l)}{d+1} + \frac{d}{d+1} b \end{aligned}$$

and similarly

(39) $$\begin{aligned} f(x_i) &= \frac{1}{J_i} f(x^l) + \frac{1}{J_i} \sum_{x_j \in R_i^l \setminus \{x^l\}} f(x_j) \\ &\geqq \frac{1}{J_i} f(x^l) + \frac{J_i - 1}{J_i} a \geqq \frac{1}{d+1} f(x^l) + \frac{d}{d+1} a. \end{aligned}$$

Hence the values of f on $R_n^{l+1} \cup R_m^{l+1}$ are all in the interval

$$[a_l, b_l] = \left[\frac{f(x^l)}{d+1} + \frac{d}{d+1}a, \ \frac{f(x^l)}{d+1} + \frac{d}{d+1}b\right] \tag{40}$$

of length $(b-a)d/(d+1)$.

Similarly, if $k > l+1$ we can find an x^{l+1} in $\bigcap_{x_j \in R_n^{l+2} \cup R_m^{l+2}} R_j^{l+1}$. This together with (40) gives by the same calculations as in (38) and (39) that

$$f(R_n^{l+2} \cup R_m^{l+2}) \subset [a_{l+1}, b_{l+1}] \tag{41}$$

where

$$[a_{l+1}, b_{l+1}] = \left[\frac{f(x^{l+1})}{d+1} + \frac{d}{d+1}a_l, \frac{f(x^{l+1})}{d+1} + \frac{d}{d+1}b_l\right] \tag{42}$$

is of length $(b-a)(d/(d+1))^2$.

Continuing inductively in this way we get that

$$f(R_n^k \cup R_m^k) \subset [a_{k-1}, b_{k-1}] \quad \text{with} \quad b_{k-1} - a_{k-1} = (d/(d+1))^{k-l}(b-a). \tag{43}$$

Since by definition the values of f in x_n and x_m are convex combinations of the values of f in R_n^k and R_m^k respectively, we get that $f(x_n)$ and $f(x_m)$ are in $[a_{k-1}, b_{k-1}]$ too and the lemma is proved. ■

LEMMA 11. $C(K) = \overline{\bigcup_{l=1}^{\infty} E_{n_l}}$.

PROOF. Let $g \in C(K)$ and $\varepsilon > 0$ be given. Let l be big enough so that

$$\rho(x,y) < 2\delta_l \quad \text{implies} \quad |g(x) - g(y)| < \varepsilon. \tag{44}$$

By Lemma 10 there exists an $f \in E_{n_l}$ such that

$$f(x_m) = g(x_m) \quad \text{for} \quad 1 \leqq m \leqq n_l. \tag{45}$$

We claim that $\|f - g\| \leqq \varepsilon$. Indeed, let i be a positive integer and let $x_m \in R_i^l$. Then by Lemma 7, $\rho(x_m, x_i) < 2\delta_l$, so by (44) and (45),

$$|f(x_m) - g(x_i)| = |g(x_m) - g(x_i)| < \varepsilon. \tag{46}$$

Since $f(x_i)$ is a convex combination of points in $f(R_i^l)$, (46) gives that

$$|f(x_i) - g(x_i)| < \varepsilon. \tag{47}$$

The sequence $\{x_i\}_{i=1}^{\infty}$ is dense in K, so we are done. ■

We are now ready to define the functions $\{e_{n,i}\}_{i \leqq n}$. This is done inductively by

$$e_{1,1} = 1_K, \tag{48}$$

(49) $$e_{n,n}(x_m) = \delta_{nm} \quad \text{for} \quad m \leqq n_{l(n)+1}, \quad n = 2, 3, \cdots,$$

(50) $$e_{n,n} \in E_{n_{l(n)+1}},$$

and

(51) $$e_{n+1,i} = e_{n,i} - e_{n,i}(x_{n+1})\, e_{n+1,n+1}, i < n + 1, \quad n = 1, 2, \cdots.$$

We also set

(52) $$E_n = \operatorname{span}\{e_{n,i}\}_{i=1}^{n}, \quad n = 1, 2, \cdots.$$

Observe that (52) agrees with the previous definition of E_{n_l} and that (51) implies that

(53) $$E_n \subset E_{n+1}, \quad n = 1, 2, \cdots.$$

From Lemma 11 we get now that

(54) $$C(K) = \overline{\bigcup_{n=1}^{\infty} E_n}.$$

We claim that

(55) $$0 \leqq e_{n,i} \leqq 1_K \quad \text{for} \quad i \leqq n \quad \text{and} \quad n = 1, 2, \cdots.$$

Indeed, (55) is clearly true for $n = 1$, so suppose it is true for all integers $\leqq n$. (49) and (50) give that (55) is true for $e_{n+1,n+1}$, so let $i \leqq n$. If $m < n + 1$ or $n + 1 < m \leqq n_{l(n+1)+1}$ then $e_{n+1,i}(x_m) = e_{n,i}(x_m)$. Moreover, $e_{n+1,i}(x_{n+1}) = 0$, so $0 \leqq e_{n+1,i}(x_m) \leqq 1$ for $1 \leqq m \leqq n_{l(n+1)+1}$. This and the fact that $e_{n+1,i} \in E_{n_{l(n+1)+1}}$ give that $0 \leqq e_{n+1,i} \leqq 1_K$. ■

The same arguments show that

(56) $$e_{n,i}(x_m) = \delta_{i,m} \quad \text{for} \quad 1 \leqq i, m \leqq n, \quad n = 1, 2, \cdots.$$

An induction argument using (48) and (51) shows easily that

(57) $$\sum_{i=1}^{n} e_{n,i} = 1_K \quad \text{for} \quad n = 1, 2, \cdots.$$

Hence $\{e_{n,i}\}_{i=1}^{n}$ is a non-negative partition of unity with $\| e_{n,i} \| = 1$, so E_n is isometric to l_n^∞. Let us show that the corresponding matrix $A = \{e_{n,i}(x_{n+1})\}_{i \leqq n}$ has the properties (16) and (17) of Theorem 2. First

(58) $$\text{For all } n,\ i \leqq n,\ e_{n,i}(x_{n+1}) \in \{0, 1/J_{n+1}\}.$$

Indeed, if $n = n_{l(n)+1}$, (58) follows at once from (56) and (35), since $e_{n,i} \in E_{n_{l(n)+1}}$. If $n < n_{l(n)+1}$ then $e_{n,n}(x_{n+1}) = 0$, so $e_{n,i}(x_{n+1}) = e_{n-1,i}(x_{n+1})$, which equals

$e_{n-2,i}(x_{n+1})$, $e_{n-3,i}(x_{n+1})$ and so on. Finally if $i \leqq n_{l(n)}$ we get $e_{n,i}(x_{n+1}) = e_{n_{l(n)},i}(x_{n+1})$. Since $e_{n_{l(n)},i} \in E_{n_{l(n)}}$, (58) follows from (56) and (35). If $i > n_{l(n)}$ then $e_{n,i}(x_{n+1})$ equals $e_{i,i}(x_{n+1})$ which is 0 by (49). This proves (16) of Theorem 2. To prove (17) we show:

(59) For each $m, l = 1, 2, \cdots$ at most $d+1$ of the numbers $\{e_{n_l,i}(x_m)\}_{i=1}^{n_l}$ are non-zero.

If $m \leqq n_l$ (59) is obvious by (56), so let $m > n_l$. If $e_{n_l,i}(x_m) > 0$, then by (56) and the definition (35) of E_{n_l}, x_i must be a relative in the lth generation of x_m, i.e. $x_i \in R_m^l$, so (59) follows from Lemma 8. This proves the "only if"-part of Theorem 2.

6. Proof of the "if"-part of Theorem 1

Assume now that K is a compact metric space and $A = \{a_{n,i}\}_{i \leq n}$ is a representing matrix for $C(K)$, so that $\Sigma_{i=1}^n a_{n,i} = 1$ and $\lambda(A) > 1/(d+2)$. We use the notation of Section 2. For $\varepsilon > 0$ set

(60) $$U_{n,i}^{\varepsilon} = \{x \in K \mid e_{n,i}(x) > \varepsilon\}.$$

We need the following lemma:

LEMMA 12. *For all* $\varepsilon > 0$, $\lim_{n\to\infty} \max_{1 \leq i \leq n} \text{diameter}(U_{n,i}^{\varepsilon}) = 0$.

PROOF. Let $\varepsilon > 0$ and assume the converse, i.e. that there are a $\delta > 0$, a subsequence $0 < n_1 < n_2 < \cdots$ of the integers, integers $i_1, i_2, \cdots$ with $i_m \leqq n_m$, and sequences $\{y_m\}_{m=1}^{\infty}$ and $\{z_m\}_{m=1}^{\infty}$ in K so that

(61) $$\rho(y_m, z_m) \geqq \delta, \quad e_{n_m,i_m}(y_m) > \varepsilon \quad \text{and} \quad e_{n_m,i_m}(z_m) > \varepsilon.$$

Passing to subsequences we may assume that the sequences $\{y_m\}_{m=1}^{\infty}$ and $\{z_m\}_{m=1}^{\infty}$ converge to y and z respectively. By (61), $y \neq z$, whence we can find an $f \in C(K)$ so that if m is large enough, then

(62) $$f(y_{n_m}) = 0 \quad \text{and} \quad f(z_{n_m}) = 1$$

and so that $0 \leqq f \leqq 1$. Let m be any integer large enough to satisfy (62). We show that $\|f - Q_{n_m} f\| \geqq \varepsilon^2/(1+\varepsilon)$ for each such m, which will contradict Lemma 5. If $\|f - Q_{n_m} f\| < \varepsilon^2/(1+\varepsilon)$, then

(63) $$\frac{\varepsilon^2}{1+\varepsilon} > |f(y_{n_m}) - Q_{n_m} f(y_{n_m})| = Q_{n_m} f(y_{n_m}) = \sum_{i=1}^{n_m} f(x_i)\, e_{n_m,i}(y_{n_m})$$
$$\geqq f(x_{i_m})\, e_{n_m,i_m}(y_{n_m}) \geqq f(x_{i_m})\, \varepsilon$$

and similarly

$$\frac{\varepsilon^2}{1+\varepsilon} > |f(z_{n_m}) - Q_{n_m}f(z_{n_m})| = 1 - \sum_{i=1}^{n_m} f(x_i)\, e_{n_m,i}(z_{n_m})$$

(64)
$$= 1 - f(x_{i_m})\, e_{n_m,i_m}(z_{n_m}) - \sum_{i \neq i_m} f(x_i)\, e_{n_m,i}(z_{n_m})$$

$$\geqq 1 - f(x_{i_m}) - \sum_{i \neq i_m} e_{n_m,i}(z_{n_m}) \geqq 1 - f(x_{i_m}) - (1-\varepsilon) = \varepsilon - f(x_{i_m}).$$

(63) and (64) together give that

(65)
$$\frac{\varepsilon^2}{1+\epsilon} > \varepsilon - \frac{\varepsilon}{1+\varepsilon} = \frac{\varepsilon^2}{1+\varepsilon}\,.$$

This proves Lemma 12. ■

Recall that by Lemma 4, if $\lambda(A) > \lambda > 1/(d+2)$, there exist infinitely many n's so that $\max_{1 \leqq i \leqq n} e_{n,i} \geqq \lambda$, so we can find a sequence $1 \leqq n_1 < n_2 < \cdots$ with

(66)
$$\max_{1 \leqq i \leqq n_l} e_{n_l,i} > \frac{1}{d+2} \quad \text{for} \quad l = 1, 2, \cdots.$$

Thus for each $l = 1, 2, \cdots, \{U_{n_l,i}^{1/(d+2)}\}_{i=1}^{n_l}$ covers K. This open cover is clearly of order $\leqq d$, since $\sum_{i=1}^{n_l} e_{n_l,i} \equiv 1$ and by Lemma 12 we have that $\lim_{l \to \infty} \operatorname{mesh}\{U_{n_l,i}^{1/(d+2)}\}_{i=1}^{n_l} = 0$. From this it follows that $\dim K \leqq d$. ■

7. Dimension and Choquet-simplices

In this section we prove an extension of the "if"-part of Theorem 1. Before stating this extension, we introduce some notations. Let S be an infinite dimensional metrizable Choquet-simplex (cf. [1]). $A(S)$ is the Banach space of continuous real valued affine functions on S with the sup-norm. As remarked in Section 1, $A(S)$-spaces can be characterized as those preduals of L_1 which have a representing matrix $A = \{a_{n,i}\}_{i \leqq n}$ with $\sum_{i=1}^{n} a_{n,i} = 1$. Every $C(K)$-space for K compact metric is isometric to the space $A(S)$, where S is the state space $S = \{\mu \in C(K)^* \mid \|\mu\| = \mu(1_K) = 1\}$ with the ω^*-topology from $C(K)^*$ ([1]). Clearly in this case $K = \partial_e S$. Conversely it is well known that if $\partial_e S$ is compact then $A(S) = C(\partial_e S)$.

Now let $A = \{a_{n,i}\}_{i \leqq n}$ with $\sum_{i=1}^{n} a_{n,i} = 1$ be a matrix representing a space $A(S)$ and $\{e_{n,i}\}_{i \leqq n}$ the corresponding unit-vector basis of E_n. We regard $A(S)$ as a closed subspace of $C(\overline{\partial_e S})$. Preceed now as in Section 2 with $\overline{\partial_e S}$ instead of K and $A(S)$ instead of $C(K)$. Problems arise only in the proof of Lemma 2, but we can also prove

LEMMA 13. *The set* $\{x_i\}_{i=1}^{\infty}$ *is contained in* $\partial_e S$ *and dense in* $\overline{\partial_e S}$.

PROOF. Let us first prove the latter. Assume the converse, i.e. that $V = \overline{\{x_i\}_{i=1}^{\infty}}$ does not contain $\partial_e S$. Then by the Krein-Milman-Rutman theorem ([2, p. 80])

$$H = \overline{\text{conv}}\, V \subsetneqq S. \tag{67}$$

In particular there is a $z \in \partial_e S \setminus H$. By the Hahn-Banach theorem there exists an $f \in A(S)$ with $f(z) = 1$ and $\max f(H) \leqq 0$. As in the proof of Lemma 2 this implies that $\text{dist}(f, \cup_{n=1}^{\infty} E_n) \geqq \frac{1}{2}$, which contradicts (1). To prove the first assertion let i be any positive integer and let

$$f = \sum_{k=0}^{\infty} 2^{-k-1} e_{i+k,i}. \tag{68}$$

Clearly

$$\|f\| \leqq 1 \tag{69}$$

and

$$\{x_i\} = \{x \in S \mid f(x) = 1\}. \tag{70}$$

Thus $x_i \in \partial_e S$, since it is a unique peak point for f. ■

THEOREM 4. *Let S be a metrizable Choquet-simplex and assume that* $A(S)$ *has a representing matrix* $A = \{a_{n,i}\}_{i \leq n}$ *with* $\sum_{i=1}^{n} a_{n,i} = 1$ *and* $\lambda(A) > 1/(d+2)$. *Then* $\dim C \leqq d$ *for every compact subset C of* $\partial_e S$.

PROOF. Proceed as in Section 7 with K and $C(K)$ replaced by C and $A(S)$ respectively. The existence of a function $f \in A(S)$ so that $0 \leqq f \leqq 1$ and with the property (62) follows easily from [1, p. 91]. ■

REMARKS. (i) We cannot prove that $\dim \partial_e S \leqq d$ by replacing in Section 6 K by $\partial_e S$. Compactness is strongly needed in Lemma 12. In the next section we give an example where diameter $(\{x \in \partial_e S \mid e_{n,1}(x) > 1/3\}) = 1$ all n. (ii) The "only if"-part of Theorem 1 cannot be extended to $A(S)$-spaces. We give in Section 8 an example of a simplex S with $\dim \partial_e S = 0$, but $\lambda(A) \leqq \frac{1}{2}$ for every matrix A representing $A(S)$.

8. Open problems and examples

From Section 7 the following problem arises:

PROBLEM 2. Let S be a metrizable Choquet simplex and $A=\{a_{n,i}\}_{i\leq n}$ a matrix with $\Sigma_{i=1}^{n} a_{n,i}=1$ and $\lambda(A)>1/(d+2)$ representing $A(S)$. Does it follow that $\dim \partial_e S \leqq d$ or even $\dim \overline{\partial_e S} \leqq d$?

Observe that from Theorem 4 it does not follow that $\dim \partial_e S \leqq d$. There exist simplices S with $\dim C<\dim \partial_e S$ for every compact subset C of $\partial_e S$ (see [6] for such examples and also [3]). The following example will justify the remarks preceding Theorem 4.

EXAMPLE 1. Let X be the subspace of c — the space of converging sequences in R — consisting of those sequences $(t_i)_{i=1}^{\infty}$ for which

(71) $$\lim_{i\to\infty} t_i=\tfrac{1}{2}(t_1+t_2).$$

Define $\{e_{n,i}\}_{i\leq n}$ in X as follows:

(72) $$e_{1,1}\equiv 1,$$

(73) $$e_{n,1}(k)=\begin{cases}1 & \text{if} \quad k=1\\ 0 & \text{if} \quad 2\leqq k\leqq n,\\ \tfrac{1}{2} & \text{else}\end{cases}$$

(74) $$e_{n,2}(k)=\begin{cases}1 & \text{if} \quad k=2\\ 0 & \text{if} \quad k=1 \quad \text{or} \quad 3\leqq k\leqq n,\\ \tfrac{1}{2} & \text{else}\end{cases}$$

(75) $$e_{n,i}(k)=\delta_{i,k} \qquad \text{else.}$$

i.e.

(76) $$e_{1,1}=(1,1,1,1,1,\cdots),$$

$$e_{2,1}=(1,0,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\cdots),\ e_{2,2}=(0,1,\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\cdots),$$

$$e_{3,1}=(1,0,0,\tfrac{1}{2},\tfrac{1}{2},\cdots),\ e_{3,2}=(0,1,0,\tfrac{1}{2},\tfrac{1}{2},\cdots),\ e_{3,3}=(0,0,1,0,0,\cdots),$$

$$e_{4,1}=(1,0,0,0,\tfrac{1}{2},\cdots),\ e_{4,2}=(0,1,0,0,\tfrac{1}{2},\cdots),$$

$$e_{4,3}=(0,0,1,0,0,\cdots),\ e_{4,4}=(0,0,0,1,0,\cdots),\ \cdots.$$

Set

(77) $$E_n=\operatorname{span}\{e_{n,i}\}_{i=1}^{n}, \quad n=1,2,\cdots$$

and let $A=\{a_{n,i}\}_{i\leq n}$ be the triangular matrix with

(78) $$a_{1,1}=1,\ a_{n,1}=a_{n,2}=\tfrac{1}{2} \text{ if } n>1 \text{ and } a_{n,i}=0 \text{ else:}$$

i.e.

(79)
$$A = \left\{ \begin{array}{l} 1 \\ \frac{1}{2}\ \frac{1}{2} \\ \frac{1}{2}\ \frac{1}{2}\ 0 \\ \frac{1}{2}\ \frac{1}{2}\ 0\ 0 \\ \frac{1}{2}\ \frac{1}{2}\ 0\ 0\ 0 \\ \cdots \end{array} \right\}.$$

It is easy to see then, that (1)–(4) are valid, so X is an $A(S)$-space with A as a representating matrix. Clearly $\overline{\partial_e S} = N$, the natural numbers, and the points defined in (11) are $x_i = i$. $\overline{\partial_e S} = \bar{N} = N \cup \{\infty\}$, the one-point compactification of N with the metric $\rho(n,m) = |(1/n) - (1/m)|((1/\infty) = 0)$. That $\infty \notin \partial_e S$ follows clearly from (71), since $\infty = \frac{1}{2}(1+2)$. We have now that

(80)
$$\lambda(A) = \limsup_{n\to\infty} \inf_{l \geq t} \max_{1 \leq i \leq n} e_{n,i}(l) = \tfrac{1}{2}.$$

Let $V_{n,i}^{1/3} = \{k \in N \mid e_{n,i}(k) > 1/3\}$. Then by (73) $V_{n,i}^{1/3} = \{1, n+1, n+2, \cdots\}$ with diameter $(V_{n,i}^{1/3}) = 1$. Hence Lemma 12 is not valid in this case. To show that the converse of Theorem 4 does not hold, we shall show that for every matrix $A = \{a_{n,i}\}_{i \leq n}$ with $\Sigma_{i=1}^{n} a_{n,i} = 1$ representing X we must have $\lambda(A) \leq \frac{1}{2}$ although $\dim \partial_e S = 0$. Indeed, let A be such a matrix, $\{e_{n,i}\}_{i \leq n}$ the corresponding partitions of unity and $\{x_i\}_{i=1}^{\infty}$ as defined in (11). Since $\partial_e S = N$ is a discrete space, it follows from Lemma 13 that $\{x_i\}_{i=1}^{\infty} = \partial_e S$. Especially we can find i_1 and i_2, so that $x_{i_1} = 1$ and $x_{i_2} = 2$, and hence, if $n \geq \max(i_1, i_2)$, we get that

(81)
$$\lim_{l\to\infty} e_{n,i_1}(l) = \tfrac{1}{2}(e_{n,i_1}(1) + e_{n,i_1}(2)) = \tfrac{1}{2}$$

and

(82)
$$\lim_{l\to\infty} e_{n,i_2}(l) = \tfrac{1}{2}(e_{n,i_2}(1) + e_{n,i_2}(2)) = \tfrac{1}{2}.$$

Hence, for every $n \geq \max(i_1, i_2)$, $\Sigma_{i \neq i_1, i_2} e_{n,i}(l) < \frac{1}{4}$ if l is big enough (use (8)). This, (81) and (82) imply that

(83)
$$\inf_{l \geq t} \max_{1 \leq i \leq n} e_{n,i}(l) \leq \tfrac{1}{2} \quad \text{for} \quad n \geq \max(i_1, i_2)$$

and we are done. ■

The following example is due to J. Lindenstrauss.

EXAMPLE 2. Let H and K be two compact metric spaces and $A = \{a_{n,i}\}_{i \leq n}$ and $B = \{b_{n,i}\}_{i \leq n}$ representing matrices for $C(H)$ and $C(K)$ respectively, so that

$$\sum_{i=1}^{n} a_{n,i} = \sum_{i=1}^{n} b_{n,i} = 1 \quad \text{and} \quad a_{n,i}, b_{n,i} \in \{0, \tfrac{1}{2}, 1\}. \tag{84}$$

We show that also $C(H \times K)$ has a representing matrix $C = \{c_{n,i}\}_{i \leq n}$ with

$$\sum_{i=1}^{n} c_{n,i} = 1 \quad \text{and} \quad c_{n,i} \in \{0, \tfrac{1}{2}, 1\}. \tag{85}$$

Indeed let $\{e_{n,i}\}_{i \leq n}$ and $\{f_{n,i}\}_{i \leq n}$ be the positive norm-one partitions of unity corresponding to the matrices A and B respectively,. Then the vectors $\{e_{n,i} \otimes f_{n,j}\}_{i,j=1}^{n}$ defined by

$$e_{n,i} \otimes f_{n,j}(h,k) = e_{n,i}(h) \cdot f_{n,j}(k) \quad (h,k) \in H \times K \tag{86}$$

constitute for each n a positive norm-one partition of unity on $H \times K$, i.e.

$$E_{n^2} = \operatorname{span}\{e_{n,i} \otimes f_{n,j}\}_{i,j=1}^{n^2} = l_{n^2}^{\infty}. \tag{87}$$

Clearly $E_{n^2} \subset E_{(n+1)^2}$ and $C(H \times K) = \overline{\bigcup_{n=1}^{\infty} E_{n^2}}$, so it remains only to show that the sequence $E_1, E_4, E_9, \cdots$ can be "filled up" so that the corresponding matrix satisfies (85). We have that for every $i, j \leq n$,

$$\begin{aligned} e_{n,i} \otimes f_{n,j} = {} & e_{n+1,i} \otimes f_{n+1,j} + a_{n,i}\, e_{n+1,n+1} \otimes f_{n+1,j} + \\ & b_{n,j}\, e_{n+1,i} \otimes f_{n+1,n+1} + a_{n,i} b_{n,j}\, e_{n+1,n+1} \otimes f_{n+1,n+1}. \end{aligned} \tag{88}$$

This shows that problems in filling up only arise when $a_{n,i} = a_{n,j} = b_{n,l} = b_{n,k} = \frac{1}{2}$. For convenience we will assume that $n = j = k = 2$ and $l = i = 1$ and fill up the gap from E_4 to E_9. The other gaps are treated similarly. The basis of E_4 is

$$e_{2,1} \otimes f_{2,1},\ e_{2,1} \otimes f_{2,2},\ e_{2,2} \otimes f_{2,1} \quad \text{and} \quad e_{2,2} \otimes f_{2,2}. \tag{89}$$

Choose as a basis for E_5 the vectors

$$\begin{gathered} e_{3,1} \otimes f_{3,2} + \tfrac{1}{2} e_{3,3} \otimes f_{3,1},\ e_{3,1} \otimes f_{3,2} + \tfrac{1}{2} e_{3,3} \otimes f_{3,2},\ e_{2,2} \otimes f_{2,1}, \\ e_{2,2} \otimes f_{2,2} \quad \text{and} \quad e_{3,1} \otimes f_{3,3} + \tfrac{1}{2} e_{3,3} \otimes f_{3,3}. \end{gathered} \tag{90}$$

Since, by (88), $e_{2,1} \otimes f_{2,1} = [e_{3,1} \otimes f_{3,1} + \frac{1}{2} e_{3,3} \otimes f_{3,1}] + \frac{1}{2}[e_{3,1} \otimes f_{3,3} + \frac{1}{2} e_{3,3} \otimes f_{3,3}]$ we get $c_{4,1} = \frac{1}{2}$ and clearly $c_{4,3} = c_{4,4} = 0$. Table I shows how we proceed

TABLE I

	1	2	3	4	5	6	7	8	9
E_4	$e_{21}\otimes f_{21}$	$e_{21}\otimes f_{22}$	$e_{22}\otimes f_{21}$	$e_{22}\otimes f_{22}$					
E_5	$e_{31}\otimes f_{31}$ $+\frac{1}{2}e_{33}\otimes f_{31}$	$e_{31}\otimes f_{32}$ $+\frac{1}{2}e_{33}\otimes f_{32}$	$e_{22}\otimes f_{21}$	$e_{22}\otimes f_{22}$	$e_{31}\otimes f_{33}$ $+\frac{1}{2}e_{33}\otimes f_{33}$				
E_6	$e_{31}\otimes f_{31}$ $+\frac{1}{2}e_{33}\otimes f_{31}$	$e_{31}\otimes f_{32}$ $+\frac{1}{2}e_{33}\otimes f_{32}$	$e_{32}\otimes f_{31}$ $+\frac{1}{2}e_{33}\otimes f_{31}$	$e_{32}\otimes f_{32}$ $+\frac{1}{2}e_{33}\otimes f_{32}$	$e_{31}\otimes f_{33}$ $+\frac{1}{2}e_{33}\otimes f_{33}$	$e_{32}\otimes f_{33}$ $+\frac{1}{2}e_{33}\otimes f_{33}$			
E_7	$e_{31}\otimes f_{31}$	$e_{31}\otimes f_{32}$ $+\frac{1}{2}e_{33}\otimes f_{32}$	$e_{32}\otimes f_{31}$	$e_{32}\otimes f_{32}$ $+\frac{1}{2}e_{33}\otimes f_{32}$	$e_{31}\otimes f_{33}$ $+\frac{1}{2}e_{33}\otimes f_{33}$	$e_{32}\otimes f_{33}$ $+\frac{1}{2}e_{33}\otimes f_{33}$	$e_{33}\otimes f_{31}$		
E_8	$e_{31}\otimes f_{31}$	$e_{31}\otimes f_{32}$	$e_{32}\otimes f_{31}$	$e_{32}\otimes f_{32}$	$e_{11}\otimes f_{33}$ $+\frac{1}{2}e_{33}\otimes f_{33}$	$e_{32}\otimes f_{33}$ $-\frac{1}{2}e_{33}\otimes f_{33}$	$e_{33}\otimes f_{31}$	$e_{33}\otimes f_{32}$	
E_9	$e_{33}\otimes f_{31}$	$e_{31}\otimes f_{32}$	$e_{32}\otimes f_{31}$	$e_{32}\otimes f_{32}$	$e_{32}\otimes f_{33}$	$e_{32}\otimes f_{33}$	$e_{33}\otimes f_{31}$	$e_{33}\otimes f_{32}$	$e_{33}\otimes f_{33}$

with the corresponding matrix elements

4	$\frac{1}{2}$	$\frac{1}{2}$	0	0				
5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0			
6	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0		
7	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0	0	
8	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0.

This observation and Theorem 2 show that if K is a product of 1 dimensional spaces then $C(K)$ has a representing matrix with $a_{i,k} \in \{0, \frac{1}{2}, 1\}$. It is easy to see that other operations (like cartesian products of infinitely many factors or disjoint unions) preserve the property $C(K)$ having a representing matrix of the above type. We are naturally led to

PROBLEM 3. Is it true that every $C(K)$-space has a representing matrix $\{a_{n,i}\}$ with $\Sigma_{i=1}^{n} a_{n,i} = 1$ and $a_{n,i} \in \{0, 1, \frac{1}{2}\}$?

The following example shows that the answer to Problem 3 is negative for $A(S)$-spaces:

EXAMPLE 3. Let X be the subspace of c consisting of those sequences $(t_i)_{i=1}^{\infty}$ for which

$$\lim_{i\to\infty} t_i = \frac{1}{3}t_1 + \frac{2}{3}t_2. \tag{91}$$

As in Example 1 it is easy to prove that X is an $A(S)$-space. We claim that X has no representing matrix $A = \{a_{n,i}\}_{i\le n}$ with $\Sigma_{i=1}^n a_{n,i} = 1$ and $a_{n,i} \in \{0, \frac{1}{2}, 1\}$. Indeed let A be any matrix with $\Sigma_{i=1}^n a_{n,i} = 1$ representing $X = A(S)$. Let $\{e_{n,i}\}_{i\le n}$ be the corresponding partitions of unity in $A(S)$ and $\{x_i\}_{i=1}^\infty$ as defined in (11). As in Example 1 we get that $\{x_i\}_{i=1}^\infty = \partial_e S = N$, so we can find i_1 and i_2 with $x_{i_1} = 1$ and $x_{i_2} = 2$. Hence if $n \geqq \max(i_1, i_2) = n_0$, we get that

$$\lim_{l\to\infty} e_{n,i_1}(l) = \frac{1}{3}e_{n,i_1}(1) + \frac{2}{3}e_{n,i_1}(2) = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 0 = \frac{1}{3}. \tag{92}$$

Let l_0 be so big that

$$e_{n_0,i_1}(l) < \frac{1}{2} \quad \text{for all} \quad l > l_0. \tag{93}$$

By the same arguments as in Example 1 we can find an $n_1 \geqq n_0$, so that $\{1, \cdots, l_0\} \subset \{x_1, \cdots, x_{n_1}\}$. Let now n be any integer $\geqq n_1$. Since $e_{n,i}(x_j) = \delta_{i,j}$ for $i, j \leqq n$ we get that $e_{n,i_1}(1) = 1$ and $e_{n,i_1}(l) = 0$ for $l = 2, \cdots, l_0$. Since (by (2)) $e_{n,i_1} \leqq e_{n_0,i_1}$ we get from (93) that $e_{n,i_1}(l) < \frac{1}{2}$ for $l > l_0$, so

$$\text{for all } n \geqq n_1\text{: } e_{n,i_1}(1) = 1 \quad \text{and} \quad e_{n,i_1}(l) < \tfrac{1}{2} \quad \text{for} \quad l > 1. \tag{94}$$

Let now $n \geqq n_1$. We have

$$e_{n,i_1} = e_{n+1,i_1} + a_{n,i_1} e_{n+1,n+1} = e_{n+1,i_1} + e_{n,i_1}(x_{n+1})\, e_{n+1,n+1}. \tag{95}$$

If $e_{n,i_1}(x_{n+1}) = 0$, $e_{n+1,i_1} = e_{n,i_1}$, so

$$\begin{aligned} e_{n+1,i_1} &= e_{n+2,i_1} + e_{n+1,i_1}(x_{n+2}) e_{n+2,n+2} \\ &= e_{n+2,i_1} + e_{n,i_1}(x_{n+2}) e_{n+2,n+2}. \end{aligned} \tag{96}$$

Continue in this way until the first $m > n_1$ for which $e_{n,i_1}(x_m) \neq 0$. Such an m exists by (92) and from (94) we get

$$0 < a_{m,i_1} = e_{n,i_1}(x_m) < \tfrac{1}{2}, \tag{97}$$

and we are done.

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